

In this chapter we are going to study three subjects, we start by a very important notion in mathematics it called logic in this section we are going to present some articles which are used in proof of theorem, proposition, corollary or lemma.

1.1 Logic : 1.1.1 Statement "Proposition" – Assertion and Predicate:

Definition 1.1.1. statement or proposition is the content of an assertion. It is either true or false, but cannot be true and false at the same time. **Example 1.1.1.** $x = 1$ is the solution of the equation $2x = 2$: it is a proposition, because this sentence is true. w is a natural number : it is not a proposition, because we cannot say this sentence is true or false. **Remark 1.1.1.** Two assertions P and R are logically equivalent (denoted $P \equiv R$) if both true or false. **Definition 1.1.2.** Predicate is a sentence which is contain some variables, if we replace each one by a value of a set, we obtain an assertion. **Example 1.1.2.** m is a divisor of 18 : it is a predicate. If we take $m=6$, we obtain that 6 is a divisor of 18 : this assertion is true. If we take $m=5$, we observe that 5 is not a divisor of 18 : this assertion is false.

1.1.2 Sentential Connectives :

In mathematical discourse and elsewhere one constantly encounters declarative sentences which have been formed by modifying a sentence with the word not or by connecting sentences with the words and, or, if ... then ... (or implies), and ... if and only if ... (or equivalent). These five words or combinations of words are called sentential connectives.

Negation :

Definition 1.1.3. In mathematics, a negation is an operator on the logical value of a proposition that sends true to false and false to true. The negation (or logical Not) of P , denoted by $\neg P$ or $P?$. **Example 1.1.3.** P : He is a student, $\neg P$: He is not a student. Q : 2 is a prime number, $\neg Q$: 2 is not a prime number.

Conjunction :

Definition 1.1.4. A logical conjunction is an operator on two logical propositions that produces a value of true if both statements are true, and false otherwise. The conjunction (or logical and) of P and Q , denoted by $P \wedge Q$. **Example 1.1.4.** (Algeria is an Africa country) and (Its mother language is English) : The composite sentence is false. (2 is an even number) and (2 is a natural number) : The composite sentence is true.

Disjunction :

Definition 1.1.5. A logical disjunction is an operator on two logical propositions that is true if either statement is true or both are true, and it is false otherwise. The disjunction (or logical or) of P and Q , denoted by $P \vee Q$. **Example 1.1.5.** (10 is an odd number) or (10 is a real number) : The composite sentence is true. (? is an integer number) or (? is a rational number) : The composite sentence is false.

Conditional :

Definition 1.1.6. The conditional connective $P \Rightarrow Q$ is a logical statement that means (if P then Q) or P implies Q . In this statement P is called the antecedent and Q is called the consequent. The composite sentence is false if the sentence P is true and the sentence Q is false, and it is true otherwise. **Example 1.1.6.** If n is an odd natural number then $2n$ is an even natural number. If $x = 1$ then $2x = 2$.

Biconditional :

Definition 1.1.7. The logical biconditional is an operator connecting two logical propositions that is true if the statements are both true or both false, and it is false otherwise. The biconditional from P to Q , denoted by $P \leftrightarrow Q$ and we read (P if and only if Q) or (P is equivalent to Q). The phrase (if and only if) is often abbreviated as (iff). **Example 1.1.7.** $2x = 2$ if and only if $x = 1$.

1.1.3 Truth Tables :

Below are truth tables for the types of composite statements, we have already discussed, as well as those for conditional and biconditional statements.

1.1.4 Tautology and Contradiction :

Tautology : **Definition 1.1.8.** A tautology is a statement which is true in every valuation of its propositional variables, independent of the truth values assigned to these variables. **Example 1.1.8.** $P \vee \neg P$

Contradiction : **Definition 1.1.9.** The negation of a tautology is a contradiction, it is

a statement which is necessarily false regardless of the truth values of its propositional variables.

Example 1.1.9. $P \rightarrow P \rightarrow \neg P \rightarrow T \rightarrow F \rightarrow F \rightarrow T \rightarrow F$ 1.1.5 Relations between statements : Theorem 1.1.1. Let A, B

and C be statements. The next equivalences are true : (a) Commutativity : $A \rightarrow B \rightarrow B \rightarrow A$ and $A \rightarrow B \rightarrow B \rightarrow A$; (b) Associativity : $A \rightarrow (B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow C$ and $A \rightarrow (B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow C$; (c) Distributive : $A \rightarrow (B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$ and $A \rightarrow (B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$; (d) $A \rightarrow (B \rightarrow A) \rightarrow A$ and $A \rightarrow (B \rightarrow A) \rightarrow A$; (e) $A \rightarrow A \rightarrow A$ and $A \rightarrow A \rightarrow A$; (f) $A \rightarrow F \rightarrow A$ and $A \rightarrow T \rightarrow A$ such that : T represent truth statement and F represent false statement; (g) $A \rightarrow T \rightarrow T$ and $A \rightarrow F \rightarrow F$; (h) $A \rightarrow (\neg A) \rightarrow T$ and $A \rightarrow (\neg A) \rightarrow F$; (i) $\neg(\neg A) \rightarrow A$; (j) De Morgan's law : $\neg(A \rightarrow B) \rightarrow (\neg A) \rightarrow (\neg B)$ and $\neg(A \rightarrow B) \rightarrow (\neg A) \rightarrow (\neg B)$; (l) $A \Rightarrow B \rightarrow (\neg A) \rightarrow B$; (m) $\neg(A \Rightarrow B) \rightarrow A \rightarrow (\neg B)$; (n) $(A \Rightarrow B) \rightarrow (B \Rightarrow C) \rightarrow (A \Rightarrow C)$; (o) $(A \rightarrow B) \rightarrow (A \Rightarrow B) \rightarrow (B \Rightarrow A)$. Proof. Using the truth tables. 1.1.6

Quantifiers : Let S be a nonempty set and x is a element in S. Let P(x) be a predicate, x is called a free variable because its value is not fixed in the sentence P(x). Universal quantifier : Definition 1.1.10. It is denoted by $\forall$ and informally read for all. The assertion $\forall x \in S : P(x)$ is true if P(x) is true for all values of x in S. Example 1.1.10. $\forall n \in \mathbb{N} : n^2 - 4 \geq 2n - 5$. Existential quantifier : Definition 1.1.11. It is denoted by $\exists$ and informally read there exists an x. The statement $\exists x \in S : P(x)$ is true if P(x) is true for at least one value of x in S. Example 1.1.11. $\exists x \in \mathbb{N} : x^2 + x - 2 = 0$. Remarks 1.1.1. (1) $\neg(\forall x \in S : P(x)) \rightarrow \exists x \in S : (\neg P(x))$. (2) $\neg(\exists x \in S : P(x)) \rightarrow \forall x \in S : (\neg P(x))$. (3) $\exists! x \in S : P(x)$, we read " there exists a unique element x in S : P(x) ", this statement is true if P(x) is true just for one element x in S. (4) We can also define predicates with multiple free variables. They can be called theorems, propositions, lemmas, corollaries and exercises.

1.1.7 Strategies for Proofs : True statements in mathematics have different names. That is a proof by contrapositive begins by assuming that Q is false (i. e., $\neg Q$ is true). Show that, the sum of rational number and irrational number is an irrational number. Z such that : $x = \frac{c}{d}$ (where $d \neq 0$), then

$y = \frac{a}{b} - \frac{c}{d}$, which is equivalent to $y = \frac{ad - bc}{bd}$, implies $\exists e, f \in \mathbb{Z}$ After produces a series of direct implications leading to the conclusion that P is false (i. e., $\neg P$ is true). Z, such that : $x + y = \frac{a}{b}$ (where $b \neq 0$), because $x \in \mathbb{Q}$, then $\exists c, d \in \mathbb{Z}$ ($e = ad - bc$ and $f = bd$) such that : $y = \frac{e}{f}$, implies $y \in \mathbb{Q}$ (b) Proof by Contrapositive : A proof by contrapositive takes advantage of the mathematical equivalence $(P \Rightarrow Q) \rightarrow (\neg Q \Rightarrow \neg P)$. It follows that Q cannot be false when P is true. (c) Proof by Contradiction : A proof by contradiction is based on the mathematical equivalence $\neg(P \Rightarrow Q) \rightarrow (P \wedge \neg Q)$. We assume that there exists x_1 and x_2 ($x_1 \neq x_2$) such that : P(x_1) and P(x_2) and we arrive that $x_1 = x_2$ (d) Proof by Counterexample : A proof by counterexample of the proposition ($P \Rightarrow Q$ is false) is based to find a particular case which is the proposition $P \Rightarrow Q$ false. Is the following proposition true or false ? There are five different proofs for prove those statements. We assume that the proposition Q is false, that means n is an even number, then there exists k $\in \mathbb{Z}$ In a proof by contradiction, we start by assuming that both P and $\neg Q$ are true. Then, a series of direct implications are given that lead to a logical contradiction. In the second section, we study set theory and finally, we are going to study the maps. Examples 1.1.1. S : P(x, y) is not equivalent to $\exists y \forall x P(x, y)$ (a) Direct Proof : The simplest form of proof for a statement of the form $P \Rightarrow Q$ is the direct proof. First assume that P is true. Produce a series of steps, each one following from the previous ones, that eventually leads to conclusion Q. Example 1.1.12. N such that : $n = 2k + 1$, implies $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, putting $k' = 2k^2 + 2k$ then $\exists k' \in \mathbb{Z}$ The proof has been completed. Remark 1.1.2. For prove a proposition as the form " $\exists x \in S : P(x)$ ", we start by : Let us $x \in S$ and we finish by P(x)

is true. n such that : $n = 2k$ which is equivalent to $n^2 = 4k^2 = 2(2k^2)$, then n^2 is an even number. The proof has been completed. Hence, $P \rightarrow Q$ cannot be true and $P \Rightarrow Q$. Example 1.1.14. Q , so we have a contradiction. Hence, $x + y \neq Q$. The proof has been completed. Remark 1.1.3. For prove that the following statement is true $\forall x \exists y$ Usually we use the proof by contradiction. (1) $\exists x \forall y (f(x) \geq 0) \rightarrow \forall x \exists y (f(x) < 0)$. (2) $\forall x \exists y (R : x^2 + 1 = 0) \rightarrow \forall x \exists y (R : x^2 + 1 \neq 0)$. (3) $\forall x \exists y (S : P(x, y)) \rightarrow \exists y \forall x (S : P(x, y))$. Attention 1.1.1. $\forall x \exists y (S : P(x, y)) \rightarrow \exists y \forall x (S : P(x, y))$. Show that, if $n \in \mathbb{N}$ is an odd number then n^2 is also. $\neg Q$. Example 1.1.15